Tutorial 2 (Jan 21, 23)

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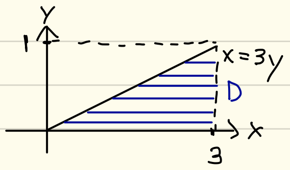
Q1) Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

Sol) Idea: Applying Fubini's Theorem (Stronger version) to interchange the order of integration.

Step 1: Describe the region of integration D using given order of variables.

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \,;\, 3y \le x \le 3\}$$

Picture:



Step 2: Describe D using different order of variables.

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 3 \,;\, 0 \le y \le \tfrac{x}{3}\}$$

Step 3: Interchange the order of integration using Fubini's Thm.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx = \int_0^3 \left[ye^{x^2}\right]_0^{\frac{x}{3}} dx$$

$$= \frac{1}{3}\int_0^3 xe^{x^2}\,dx$$

$$= \frac{1}{6}\left[e^{x^2}\right]_0^3$$

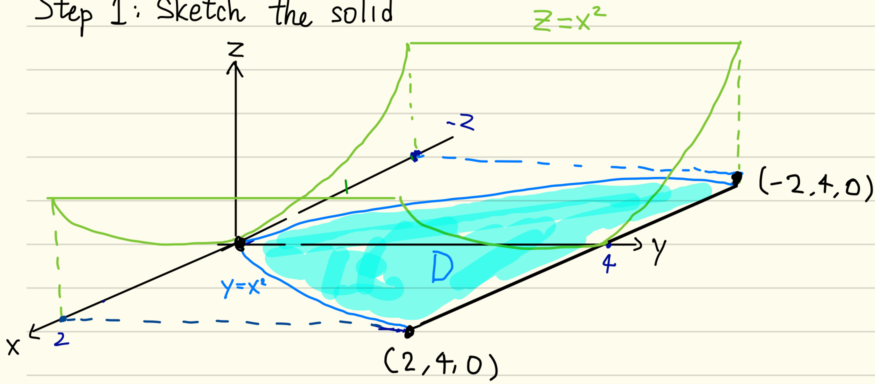
$$= \frac{1}{6}(e^9 - 1)$$

Q2) Find the volume of the solid bounded by the cylinders

$z=x^2$, $y=x^2$ and the planes $z=0$, $y=4$.

Sol) Idea: Draw some figure to help determining the double integral.

Step 1: Sketch the solid



Step 2: Describe the region of integration D.

$$D = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2 ;\ x^2 \le y \le 4\}$$

Step 3: Compute the volume by Fubini's Theorem.

$$\text{Volume} = \iint_D x^2\, dA = \int_{-2}^{2}\int_{x^2}^{4} x^2\, dy\, dx \quad \text{(By Fubini's Thm)}$$

$$= \int_{-2}^{2} \left[x^2 y\right]_{x^2}^{4} dx = \int_{-2}^{2} (4x^2 - x^4)\, dx = 2\int_0^2 (4x^2 - x^4)\, dx$$

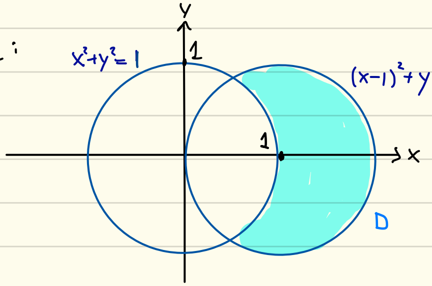
$$= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = 2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{128}{15}$$

Q3) Find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

Sol) Idea: Determine the polar coordinate description of the region.

Step 1: Sketch the region D.

Picture:

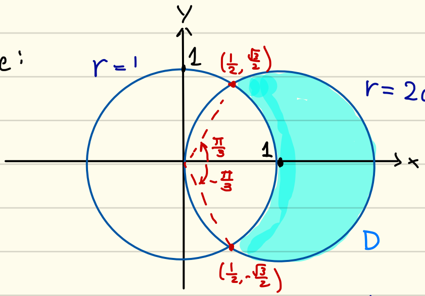


Step 2: Describe D in terms of polar coordinates.

Put $x=r\cos\theta,\ y=r\sin\theta$: (where $r>0,\ -\pi\leq\theta<\pi$)

$$\begin{cases} x^2+y^2=1 \iff r=1 \\ (x-1)^2+y^2=1 \iff r=2\cos\theta \end{cases}$$

Picture:



$$D=\left\{(r,\theta)\in(0,+\infty)\times[-\pi,\pi)\ \middle|\ -\frac{\pi}{3}\leq\theta\leq\frac{\pi}{3},\ 1\leq r\leq 2\cos\theta\right\}$$

Step 3: Evaluate the area by a double integral using polar coordinates.

$$\text{Area} = \iint_D 1\,dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(4\cdot\left(\frac{1+\cos 2\theta}{2}\right) - 1\right) d\theta$$

$$= \int_0^{\frac{\pi}{3}} (1 + 2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$